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Ordinary differential equations of probability functions of convoluted distributions

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ABSTRACT

Convolution is the sum of independent and identically distributed random variables. Derivatives of the probability density function (PDF) of probability distribution often lead to the construction of ordinary differential equation whose solution is the PDF of the given distribution. Little have been done to extend the construction of the ODE to the PDF, quantile function (QF), survival function (SF), hazard function (HF) and the reversed hazard function (RHF) of convoluted probability distributions. In this paper, three probability distributions were considered namely: Constant parameter convoluted exponential distribution (CPCED), convoluted uniform exponential distribution (CUED) and different parameter convoluted exponential distribution (DPCED). First order ordinary differential equations whose solutions were the PDF, SF, HF and RHF for the probability functions of CPCED by the use of differential calculus. The case of the QF was second order nonlinear differential equations obtained by the use of Quantile Mechanics. Similarly, the same was obtained for CUED for the two cases of the distribution. Some new relationships were obtained for the PDF, SF and HF, and also the RHF, PDF and CDF with their corresponding first derivatives. The difficulty of obtaining the ODE for the probability functions of the DPCED was due to the different parameters that characterize the distribution. The use of partial different equations is not an alternate because the distribution has only one independent variable.

1. Introduction

The concept of convolution is the sum of independent and identically distributed random variables and the construction of linear combinations of random variables. The primary rationale of convolution is to that the Probability Density Function (PDF) or Probability mass function (PMF) of a sum of a given random variables is the convolution of their corresponding PDF or PMF respectively. The second rationale is to determine if the convoluted random variables is more flexible than the parent random variables. The general formula for the sum of two discrete and continuous random variables ($Z = X + Y$) are given as:

$$P(Z = z) = \sum_{-\infty}^{\infty} P(X = r)P(Y = z - r) \quad (1)$$

$$f_z(Z) = \int_{-\infty}^{\infty} f_Y(z - x)f_X(x)dx = \int_{-\infty}^{\infty} f_X(z - y)f_Y(y)dy \quad (2)$$

In distribution theory, several convolutions are available and been studied over the years. It is well known that generally that the following are true: The sum of Bernoulli random variables is Binomial random variable, the sum of geometric random variables is the negative binomial random variable and the sum of exponential random variables is the gamma random variable, sum of Bernoulli random variables is Binomial random variable and so on.

In particular, several convolutions have been obtained. Examples are summarized in Table 1.

Differential calculus is routinely applied to the PDF to obtain the mode of any given probability distribution. In the same vein, differentiation can be applied to the PDF to obtain ordinary differential equation (ODE), whose solutions converge to the original PDFs. Some of the available are listed:

1. Chi-square distribution

$$2xf'(x) + (-k + x + 2)f(x) = 0 \quad (3)$$

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2. Rayleigh distribution

$$\sigma^2 x f'(x) + (x^2 - \sigma^2) f(x) = 0 \quad (4)$$

3. Inverse- Gamma distribution

$$x^2 f'(x) + (\alpha x - \beta + x) f(x) = 0 \quad (5)$$

4. F distribution

$$2x(d_1 x + d_2) f'(x) + (2d_1 x + d_1 d_2 x - d_1 d_2 + 2d_2) f(x) = 0 \quad (6)$$

5. Laplace distribution

$$f'(x) + f(x) = 0 \quad if \quad x \geq \mu \quad (7)$$
$$bf'(x) - f(x) = 0 \quad if \quad x < \mu \quad (8)$$

Table 1: Authors contributions to the convoluted PDF literature

Convoluted Probability Density	Author(s)
Exponential distribution	Oguntunde et al. (2014); Owoloko et al. (2016)
Beta Weibull distribution	Nadarajah and Kotz (2006); Sun (2011)
Beta exponential	Shittu et al. (2012)
Compound Geometric	Willmot and Cai (2004)
Normal	Bromiley (2003)
Poisson	Samaniego (1976)

This was contained in the textbook jointly authored by Johnson et al. (1994).

6. Beta Distribution

$$x(x-1) f'(x) + (\alpha - 1 - (\alpha + \beta - 2)x) f(x) = 0 \quad (9)$$

This was pioneered by Elderton (1906).

7. Raised Cosine distribution

$$2s^3 f''(x) - 2\pi^2 s f(x) + \pi^2 = 0 \quad (10)$$

This was retrieved from Rinne (2010) homepage on MATLAB Tutorials.

8. Arcsine distribution

$$2(x-1) x f'(x) + (2x - 1) f(x) = 0 \quad (11)$$

9. Lomax distribution

$$(\lambda + x) f'(x) + (\alpha + 1) f(x) = 0 \quad (12)$$

The details are contained in Balakrishna and Lai (2009)

10. Beta Prime distribution or Inverted beta distribution

$$(x^2 + x) f'(x) + (-\alpha + \beta x + x + 1) f(x) = 0 \quad (13)$$

Culled from the book authored by Johnson et al. (1995).

11. Gamma distribution

$$\beta x f'(x) + (-\alpha\beta + \beta + x) f(x) = 0 \quad (14)$$
$$x f'(x) + (-k + \theta x + 1) f(x) = 0 \quad (15)$$

12. Rice distribution

$$\sigma^4 x^2 f''(x) + (2\sigma^2 x^3 - \sigma^4 x) f'(x) + (\sigma^4 - v^2 x^2 + x^4) f(x) = 0 \quad (16)$$

13. Nagakami distribution

$$x`\Omega f'(x) + (2mx^2 - 2m`\Omega + `\Omega) f(x) = 0 \quad (17)$$

14. Levy distribution

$$2(x-\mu)^2 f'(x) + (3x - c - 3\mu) f(x) = 0 \quad (18)$$

15. Log-Laplace distribution

$$bxf'(x) + (b-1) f(x) = 0 \quad if \quad x < \mu \quad (19)$$
$$bxf'(x) + (b+1) f(x) = 0 \quad if \quad x < \mu \quad (20)$$

But the cases of ODE of convoluted probability distributions and mixture of exponential distribution and uniform distribution have not been reported in scientific literature to the best of our knowledge. Furthermore, the ODE of quantile function (QF), survival function (SF), hazard function (HF) and reversed hazard function (RHF) of convoluted probability distributions has not been studied. Again, the Quantile Mechanics approach of obtaining the nonlinear ODE of quantile functions has not been applied to either convoluted PDF or mixture of exponential and uniform distributions. The readers are invited to see the following for details on Quantile mechanics: Steinbrecher and Shaw (2008), Kleefeld and Brazauskas (2012), and Shaw et al. (2014).

The aim is to obtain the ordinary differential equations of the probability functions to address the gaps aforementioned in the last paragraph. In summary, the list of objectives of this paper is listed:

- Obtain the ODEs whose solutions are the PDF, SF, HF and RHF of the constant parameter convoluted exponential distribution by the use of differentiation.
- Obtain the ODE whose solution is the Quantile function (QF) of the constant parameter convoluted exponential distribution by the use of Quantile Mechanics.
- Obtain the ODEs whose solutions are the PDF and SF of the two cases of the convolution of Uniform and exponential distributions by the use of differentiation.
- Obtain the ODEs whose solutions are the QF of the two cases of the convolution of Uniform and

exponential distributions by the use of differentiation.
- Establish relationship linking the PDF, SF and HF with their respective derivatives using the HF of the convoluted Uniform exponential distribution as an instance.
- Establish relationship linking the PDF, cumulative distribution function (CDF) and RHF with their respective derivatives using the RHF of the convoluted Uniform exponential distribution as an instance.
- Comment on the difficulty of obtaining the ODE of probability functions of different parameter convoluted exponential distribution.

2. Methodology

The differential calculus was used to obtain the various ODEs. The quantile mechanics approach was used to obtain the nonlinear second order ODEs of the quantile functions of the constant parameter convoluted exponential and convoluted uniform exponential distributions.

3. Results

This section contains the various results obtained by the application of the stated methodology.

3.1. Constant parameter convoluted exponential distribution (CPCED)

Definition 1: Let X and Y be independent exponentially distributed random variables with constant parameter $f(x)=\lambda e^{-\lambda x}$ and $f(y)=\lambda e^{-\lambda y}$ respectively, for $x,y\geq 0,\lambda>0$. Then the PDF of the convolution $Z = X + Y$ is given by:

$$f_Z(z)=\int_0^z f_Y(z-x)f_X(x)dx \tag{21}$$
$$f_Z(z)=\int_0^z \lambda e^{-\lambda(z-x)}\lambda e^{-zx}dx=\int_0^z \lambda^2 e^{-\lambda z}dx \tag{22}$$
$$f_Z(z)=\lambda^2 z e^{-\lambda z} \qquad 0<z<\infty \tag{23}$$

Proposition 1: The ODE whose solution is the PDF of CPCED is given by

$$zf'(z)+(\lambda z-1)f(z)=0$$

Proof: The first derivative of Eq. 23 is given as:

$$\frac{df(z)}{dz}=\lambda^2 e^{-\lambda z}-\lambda^3 e^{-\lambda z} \tag{24}$$

simplify using Eq. 23:

$$f'(z)=\frac{f(z)}{z}-\lambda f(z). \tag{25}$$

The first order ODE whose solution is the PDF of CPCED is given by:

$$zf'(z)+(\lambda z-1)f(z)=0. \tag{26}$$

Definition 2: Let X and Y be independent exponentially distributed random variables with constant parameter $f(x)=\lambda e^{-\lambda x}$ and $f(y)=\lambda e^{-\lambda y}$ respectively, for $x,y\geq 0,\lambda>0$. Then the CDF of the convolution $Z = X + Y$ is given by:

$$F(z)=1-\lambda z e^{-\lambda x}-e^{-\lambda x}. \tag{27}$$

Proposition 2: The ODE whose solution is the QF $(w(p))$ of CPCED is given by:

$$ww''(p)-(\lambda w-1)\left(w'(p)\right)^2=0.$$

Proof: Applying the Quantile mechanics approach to the PDF of CPCED to obtain. That is Eq. 23 becomes:

$$w(p)=\lambda^2 w e^{-\lambda w} \tag{28}$$
$$\frac{dw(p)}{dp}=\lambda^{-2}w^{-1}e^{\lambda w} \tag{29}$$

differentiate to obtain a second order nonlinear ODE:

$$\frac{d^2w(p)}{dp^2}=\lambda^{-2}\left[w^{-1}e^{\lambda w}\left(\frac{dw}{dp}\right)-w^{-2}e^{\lambda w}\left(\frac{dw}{dp}\right)\right] \tag{30}$$
$$\frac{d^2w(p)}{dp^2}=\left(\frac{\lambda w-1}{w}\right)\left(\frac{dw}{dp}\right)^2 \tag{31}$$

the ODE whose solution is the QF of CPCED is given by:

$$(\lambda w-1)\left(w'(p)\right)^2=0 \tag{32}$$

with the conditions:

$$(\lambda w-1)\left(w'(p)\right)^2=0.$$

This approach is a better option than finding the quantile function from Eq. 27. The resulting derivative and ODE may be difficult to obtain

Proposition 3: The ODE whose solution is the SF of CPCED is given by:

$$(\lambda z+1)s'(z)+\lambda^2 s(z)=0.$$

Proof: The survival function of CPCED is:

$$s(z)=\lambda z e^{-\lambda z}+e^{-\lambda z} \tag{33}$$

differentiate Eq. 33:

$$\frac{ds(z)}{dz}=-\lambda^2 z e^{-\lambda z} \tag{34}$$

Eq. 33 can also be written as;

$$e^{-\lambda z}=\frac{s(z)}{\lambda z+1} \tag{35}$$

substitute Eq. 35 in Eq. 34:

$$\frac{ds(z)}{dz}=-\frac{\lambda^2 z s(z)}{\lambda z+1}. \tag{36}$$

The ODE whose solution is the SF of CPCED is given by:

$$(\lambda z+1)s'(z)+\lambda^2 s(z)=0 \qquad s(0)=1 \tag{37}$$

Proposition 4: The ODE whose solution is the HF of CPCED is given by

$z(\lambda z+1)h'(z)-h(z)=0.$

Proof: The hazard function of CPCED is;

$$h(z)=\frac{\lambda^2 z}{\lambda z+1} \tag{38}$$

differentiate E. 38:

$$\frac{dh(z)}{dz}=\frac{\lambda^2 z}{\lambda z+1}-\frac{\lambda^3 z}{(\lambda z+1)^2} \tag{39}$$

substitute Eq. 38 in Eq. 39:

$$\frac{dh(z)}{dz}=\frac{h(z)}{z}-\frac{\lambda h(z)}{(\lambda z+1)} \tag{40}$$

The ODE whose solution is the HF of CPCED is given by

$$z(\lambda z+1)h'(z)-h(z)=0. \qquad h(0)=1 \tag{41}$$

Proposition 5: The ODE whose solution is the RHF of CPCED is given by

$zv'(z)+(\lambda z-1)v(z)-zv^2(z)=0.$

Proof: The reversed hazard function of CPCED is;

$$v(z)=\frac{\lambda^2 z e^{-\lambda z}}{1-\lambda^2 z e^{-\lambda z}-e^{-\lambda z}} \tag{42}$$

$$v(z)=\frac{\lambda^2 z}{e^{\lambda z}-\lambda z-1} \tag{43}$$

differentiate Eq. 43:

$$v'(z)=\left[\frac{1}{z}-\frac{\lambda(e^{\lambda z}-1)}{e^{\lambda z}-\lambda z-1}\right]v(z) \tag{44}$$

Eq. 43 can be written as:

$$\frac{v(z)}{\lambda z}=\frac{\lambda}{e^{\lambda z}-\lambda z-1} \tag{45}$$

substitute Eq. 45 in Eq. 44:

$$v'(z)=\left[\frac{1}{z}-\frac{(e^{\lambda z}-1)v(z)}{\lambda z}\right]v(z) \tag{46}$$

Eq. 43 can be written as:

$$\left(e^{\lambda z}-\lambda z-1\right)v(z)=\lambda^2 z \tag{47}$$

$$e^{\lambda z}-1=\frac{\lambda^2 z+\lambda z v(z)}{v(z)} \tag{48}$$

substitute Eq. 48 into Eq. 46 to obtain:

$$v'(z)=\left[\frac{1}{z}-\frac{\lambda^2 z+\lambda z v(z)}{\lambda z}\right]v(z) \tag{49}$$

$$v'(z)=\left[\frac{1}{z}+v(z)-\lambda\right]v(z). \tag{50}$$

The ODE whose solution is the RHF of CPCED is given by:

$$zv'(z)+(\lambda z-1)v(z)-zv^2(z)=0. \tag{51}$$

With the condition:

$v(1)=\frac{\lambda^2}{e^{\lambda}-\lambda-1}.$

3.2. Convoluted Uniform Exponential Distribution (CUED)

Definition 3: Let X be independent exponentially distributed random variables; $f(x)=\lambda e^{-\lambda x}$and Y has uniform distribution; U (0, 1). Then the PDF of the convolution Z = X + Y is given by:

$$(z)=\begin{cases}1-e^{-\lambda z}, & 0\le z<1\\ (e^{\lambda}-1)e^{-\lambda z}, & 1\le z<\infty\end{cases} \tag{52}$$

To obtain the different ODEs of the probability functions of CUED, the two cases that characterize the distribution was considered separately.

Proposition 6: The ODE whose solutions are the PDF of CUED are given by $f'(z)+\lambda f(z)-\lambda=0$ and $f'(z)+\lambda f(z)=0$ respectively.

Proof: Case I

$$f(z)=1-e^{-\lambda z}. \tag{53}$$

The first derivative of Eq. 53 is given as:

$$\frac{df(z)}{dz}=\lambda e^{-\lambda z} \tag{54}$$

simplify using Eq. 53:

$$f'(z)+\lambda f(z)-\lambda=0 \qquad f(0)=0 \tag{55}$$

Eq. 55 is the first order ODE whose solutions are the PDF of CUED for case I.

Case II:

$$f(z)=(e^{\lambda}-1)e^{-\lambda z}. \tag{56}$$

The first derivative of Eq. 56 is given as:

$$\frac{df(z)}{dz}=-\lambda(e^{\lambda}-1)e^{-\lambda z} \tag{57}$$

simplify using Eq. 56:

$$f'(z)+\lambda f(z)=0. \qquad f(0)=0 \tag{58}$$

Eq. 58 is the first order ODE whose solutions are the PDF of CUED for case II.

Proposition 7: The ODE whose solutions are the QF $(w(p))$ of CUED are given by:

$$\frac{d^2w(p)}{dp^2}+\lambda\left(\frac{dw}{dp}-1\right)\left(\frac{dw}{dp}\right)^2=0$$

and

$$\frac{d^2w(p)}{dp^2}-\lambda\left(\frac{dw}{dp}\right)^2=0.$$

Proof: Applying the Quantile mechanics approach to the PDF of CUED to obtain

Case I: Eq. 53 becomes:

$$w(p) = 1 - e^{-\lambda w} \tag{59}$$
$$\frac{dw(p)}{dp} = (1 - e^{-\lambda w})^{-1} \tag{60}$$

Differentiate to obtain a second order nonlinear ODE.

$$\frac{d^2w(p)}{dp^2} = -\lambda e^{-\lambda w}(1 - e^{-\lambda w})^{-2}\left(\frac{dw}{dp}\right) \tag{61}$$
$$\frac{d^2w(p)}{dp^2} = -\lambda e^{-\lambda w}\left(\frac{dw}{dp}\right)^3 \tag{62}$$

Eq. 60 can be written as:

$$(1 - e^{-\lambda w})\frac{dw(p)}{dp} = 1 \tag{63}$$

simplify Eq. 63 to obtain:

$$e^{-\lambda w} = \frac{\frac{dw(p)}{dp} - 1}{\frac{dw(p)}{dp}} \tag{64}$$

substitute Eq. 64 into Eq. 62 to obtain:

$$\frac{d^2w(p)}{dp^2} = -\lambda\left(\frac{dw}{dp} - 1\right)\left(\frac{dw}{dp}\right)^2. \tag{65}$$

The second order nonlinear ODE whose solution is the QF of CUED for case I is given as;

$$\frac{d^2w(p)}{dp^2} + \lambda\left(\frac{dw}{dp} - 1\right)\left(\frac{dw}{dp}\right)^2 = 0. \tag{66}$$

Case II: Eq. 56 becomes:

$$w(p) = (e^{\lambda} - 1)e^{-\lambda w} \tag{67}$$
$$\frac{dw(p)}{dp} = \frac{1}{e^{\lambda} - 1}e^{\lambda w} \tag{68}$$

Differentiate to obtain a second order nonlinear ODE.

$$\frac{d^2w(p)}{dp^2} = \frac{\lambda}{e^{\lambda} - 1}e^{\lambda w}\left(\frac{dw}{dp}\right) \tag{69}$$
$$\frac{d^2w(p)}{dp^2} = \lambda\left(\frac{dw}{dp}\right)^2 \tag{70}$$

The second order nonlinear ODE whose solution is the QF of CUED for case II is given as;

$$\frac{d^2w(p)}{dp^2} - \lambda\left(\frac{dw}{dp}\right)^2 = 0 \tag{71}$$

with the conditions:

$w(0) = 0, w'(0) = 1.$

Definition 4: Let X be independent exponentially distributed random variables; $f(x) = \lambda e^{-\lambda x}$ and Y has uniform distribution; $U\,(0,1)$.Then the CDF of the convolution $Z = X + Y$ is given by:

$$F(z) = \begin{cases} z + \frac{e^{-\lambda z}}{\lambda} - \frac{1}{\lambda}, & 0 \le z < 1 \\ \frac{(e^{\lambda} - 1)}{\lambda}\left[e^{\lambda} - e^{-\lambda z}\right], & 1 \le z < \infty \end{cases} \tag{72}$$

Proposition 8: The ODE whose solutions are the SF of CUED is given by $s'(z) + \lambda s(z) + \lambda z - \lambda = 0$ and $s'(z) + \lambda s(z) + A(\lambda) = 0$ respectively.

Proof:

Case I: The survival function is:

$$s(z) = 1 - \left(z + \frac{e^{-\lambda z}}{\lambda} - \frac{1}{\lambda}\right). \tag{73}$$

The first derivative of Eq. 73 is given as:

$$s'(z) = -(1 - e^{-\lambda z}) \tag{74}$$

Eq. 73 can be written as:

$$e^{-\lambda z} = \lambda - \lambda s(z) - \lambda z + 1 \tag{75}$$

substitute Eq. 75 into Eq. 74 to obtain:

$$s'(z) = -(1 - \lambda + \lambda s(z) + \lambda z - 1). \tag{76}$$

The first order ODE whose solutions are the SF of CUED for case I is given as;

$$s'(z) + \lambda s(z) + \lambda z - \lambda = 0 \tag{77}$$

with condition:

$s(0) = 1.$

Case II: The survival function is:

$$s(z) = 1 - \left(\frac{e^{\lambda} - 1}{\lambda}\right)(e^{\lambda} - e^{-\lambda z}). \tag{78}$$

The first derivative of Eq. 78 is given as:

$$s'(z) = -(e^{\lambda} - e^{-\lambda z})e^{-\lambda z} \tag{79}$$

Eq. 78 can be written as:

$$(e^{\lambda} - 1)(e^{\lambda} - e^{-\lambda z}) = \lambda(1 - s(z)) \tag{80}$$
$$e^{\lambda}(e^{\lambda} - 1) - e^{-\lambda z}(e^{\lambda} - 1) = \lambda(1 - s(z)) \tag{81}$$

substitute Eq. 79 into Eq. 81 to obtain:

$$e^{\lambda}(e^{\lambda} - 1) + s'(z) = \lambda(1 - s(z)) \tag{82}$$

Simplify:

$$s'(z) + \lambda s(z) + e^{\lambda}(e^{\lambda} - 1) - \lambda = 0. \tag{83}$$

A constant was introduced in order to further simplify Eq. 83:

$$A(\lambda) = e^{\lambda}(e^{\lambda} - 1) - \lambda. \tag{84}$$

Substitute Eq. 84 into Eq. 83 to obtain the first order ODE whose solution is the survival function of CUED for case II. This is given as;

$$s'(z) + \lambda s(z) + A(\lambda) = 0 \tag{85}$$

with the condition: $s(0)=1-\frac{\left(e^{\lambda}-1\right)^2}{\lambda}$

Proposition 9: The ODE whose solution is the HF of CUED is given by;

$$h'(z)=\left[\frac{f'(z)}{f(z)}+\frac{f(z)}{s(z)}\right]h(z).$$

Proof:

Case I: The hazard function is:

$$h(z)=\frac{1-e^{-\lambda z}}{1-\left(z+\frac{e^{-\lambda z}}{\lambda}-\frac{1}{\lambda}\right)} \quad (86)$$

The first derivative of Eq. 86 is given as:

$$h'(z)=\left[\frac{\lambda e^{-\lambda z}}{1-e^{-\lambda z}}+\frac{(1-e^{-\lambda z})\left(1-\left(z+\frac{e^{-\lambda z}}{\lambda}-\frac{1}{\lambda}\right)\right)^{-2}}{\left(1-\left(z+\frac{e^{-\lambda z}}{\lambda}-\frac{1}{\lambda}\right)\right)^{-1}}\right]h(z) \quad (87)$$

$$h'(z)=\left[\frac{\lambda e^{-\lambda z}}{1-e^{-\lambda z}}+\frac{(1-e^{-\lambda z})}{1-\left(z+\frac{e^{-\lambda z}}{\lambda}-\frac{1}{\lambda}\right)}\right]h(z) \quad (88)$$

The values of the probability functions are substituted into Eq. 88 to obtain:

$$h'(z)=\left[\frac{f'(z)}{f(z)}+\frac{f(z)}{s(z)}\right]h(z). \quad (89)$$

The ODE is a combination of PDF, SF and HF. To obtain the ODE of equation that is a function of HF only may be cumbersome.

Corollary: Further simplifications of Eq. 89 can be obtained such as:

$$\frac{h'(z)}{h(z)}=\frac{f'(z)}{f(z)}+\frac{f(z)}{s(z)} \quad (90)$$

$$\frac{h'(z)}{h(z)}=\frac{f'(z)}{f(z)}+h(z) \quad (91)$$

$$h'(z)-h^2(z)-\left(\frac{f'(z)}{f(z)}\right)h(z)=0 \quad (92)$$

Recalled that $s'(z)=-f(z)$, substitute in Eq. 90:

$$\frac{h'(z)}{h(z)}=\frac{f'(z)}{f(z)}-\frac{s'(z)}{s(z)} \quad (93)$$

Eqs. 89 to 93 can also be applied to all continuous probability distributions provided that the derivatives of PDF, HF and SF exist and are not equal to zero. Eq. 87 was a direct product of Eq. 90, this can be shown as follows;

Given a function:

$$h(z)=A(z)B(z). \quad (94)$$

Find the derivative of Eq. 94 by the use of the product rule of differentiation;

$$h'(z)=A(z)\,B'(z)+\,A'(z)B(z) \quad (95)$$

From Eq. 94:

$$A(z)=\frac{h(z)}{B(z)}\,, B(z)=\frac{h(z)}{A(z)} \quad (96)$$

substitute Eq. 96 into Eq. 95 to obtain;

$$h'(z)=\left(A(z)\,\frac{A'(z)}{A(z)}+\frac{B'(z)}{B(z)}\right)h(z). \quad (97)$$

Let $h(z)=\frac{f(z)}{s(z)}$, which imply that

$$A(z)=f(z), B(z)=(s(z))^{-1}, A'(z)=f'(z), B'(z)=f(z)(s(z))^{-2} \quad (98)$$

substitute Eq. 98 into Eq. 97 to obtain Eq. 89.

Case II: The hazard function is given as;

$$h(z)=\frac{(e^{\lambda}-1)e^{-\lambda z}}{1-\left(\frac{e^{\lambda}-1}{\lambda}\right)(e^{\lambda}-e^{-\lambda z})} \quad (99)$$

The same as in case I.

Proposition 10: The ODE whose solution is the RHF of CUED is given by

$$v'(z)=\left(\frac{f'(z)}{f(z)}-\frac{f(z)}{F(z)}\right)v(z).$$

Proof:

Case I: The reversed hazard function is:

$$v(z)=\frac{1-e^{-\lambda z}}{\left(z+\frac{e^{-\lambda z}}{\lambda}-\frac{1}{\lambda}\right)} \quad (100)$$

Reversed hazard function is the ratio of the PDF to the CDF of a given distribution.

Let $=v(z)=\frac{f(z)}{F(z)}$, which imply that:

$$A(z)=f(z), B(z)=(F(z))^{-1}, A'(z)=f'(z), B'(z)=-f(z)\left(F(z)\right)^{-2}. \quad (101)$$

Apply the same process of Eqs. 94 to 97 using Eq. 101 to obtain:

$$v'(z)=\left(\frac{f'(z)}{f(z)}-\frac{f(z)(F(z))^{-2}}{(F(z))^{-1}}\right)v(z) \quad (102)$$

$$v'(z)=\left(\frac{f'(z)}{f(z)}-\frac{f(z)}{F(z)}\right)v(z) \quad (103)$$

Corollary: Further simplifications of Eq. 103 can be obtained such as:

$$\frac{v'(z)}{v(z)}=\frac{f'(z)}{f(z)}-\frac{f(z)}{F(z)} \quad (104)$$

$$\frac{v'(z)}{v(z)}=\frac{f'(z)}{f(z)}-v(z) \quad (105)$$

$$v'(z)=\frac{f'(z)}{F(z)}-v^2(z) \quad (106)$$

Case II: The reversed hazard function is:

$$v(z)=\frac{\lambda e^{-\lambda z}}{e^{\lambda}-e^{-\lambda z}}. \quad (107)$$

Differentiate Eq. 107:

$$v'(z)=\left[-\frac{\lambda e^{-\lambda z}}{e^{-\lambda z}}-\frac{\lambda e^{-\lambda z}\left(e^{\lambda}-e^{-\lambda z}\right)^{-2}}{\left(e^{\lambda}-e^{-\lambda z}\right)^{-1}}\right]v(z) \quad (108)$$

$$v'(z) = -\left[\lambda + \frac{\lambda e^{-\lambda z}}{e^{\lambda} - e^{-\lambda z}}\right] v(z) \quad (109)$$

$$v'(z) = -(\lambda + v(z))v(z) \quad (110)$$

The first order ODE whose solution is the RHF of the CUED for case II is given as;

$$v'(z) + \lambda v(z) + v^2(z) = 0 \quad (111)$$

3.3. Different parameter convoluted exponential distribution (DPCED)

Definition 5: Let X and Y be independent exponentially distributed random variables with different parameter $f(x) = \sigma_1 e^{-\sigma_1 x}$ and $(y) = \sigma_2 e^{-\sigma_2 y}$ respectively, for $, y \geq 0, \sigma_1, \sigma_2 > 0$. Then the PDF of the convolution $Z = X + Y$ is given by;

$$(z) = \frac{\sigma_1 \sigma_2}{\sigma_1 - \sigma_2}(e^{-\sigma_1 z} - e^{-\sigma_1 z}) \quad (112)$$

Then the CDF of the convolution Z = X + Y is given by:

$$F(z) = 1 + \left(\frac{\sigma_1}{\sigma_2 - \sigma_1}\right) e^{-\sigma_2 z} - \left(\frac{\sigma_2}{\sigma_2 - \sigma_1}\right) e^{-\sigma_1 z} \quad (113)$$

The difficulty in obtaining the ODE whose solution is the PDF of DPCED may not be unconnected with the presence of different parameters that characterize the probability distribution. As such case, the partial differential equations (PDE) is not an alternative because of the presence of only one independent variable that characterizes the probability distribution.

4. Conclusion

First order ordinary differential equations whose solutions were the PDF, SF, HF and RHF for the probability functions of constant parameter convoluted exponential distribution by the use of differential calculus. The case of the QF was second order nonlinear differential equations obtained by the use of Quantile Mechanics. Similarly, the same was obtained for the convoluted uniform exponential distribution for the two cases of the distribution. Some new relationships were obtained for the PDF, SF and HF, and also the RHF, PDF and CDF with their corresponding first derivatives. The difficulty of obtaining the ODE for the probability functions of the different parameter convoluted exponential distribution was due to the different parameters that characterize the distribution. In addition, similar results are available for Chi-square distribution (Okagbue et al., 2017a; 2017b).

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